Coherent states for the quantum mechanics on a torus

K. Kowalski and J. Rembieliński

Department of Theoretical Physics, University of Łódź,

ul. Pomorska 149/153, 90-236 Łódź, Poland

Abstract

The coherent states for the quantum mechanics on a torus and their basic properties are discussed.

PACS numbers: 02.20.Sv, 03.65.-w, 03.65.Sq

I. INTRODUCTION

Coherent states have attracted much attention in different branches of physics [1]. In spite of their importance the theory of coherent states in the case when the configuration space has nontrivial topology is far from complete. For example the coherent states for a quantum particle on a circle [2, 3] and the sphere [4] have been introduced very recently. For an excellent review of quantum mechanics on a circle including the coherent states we refer to very recent paper [5]. We remark that is no general method for construction of coherent states for a particle on an arbitrary manifold. As a matter of fact, there exists a general algorithm introduced by Perelomov [6] of construction of coherent states for homogeneous spaces X which are quotiens: $X = G/H$ of a Lie-group manifold G and the stability subgroup H. Unfortunately, in many cases interesting from the physical point of view such as a particle on a circle, sphere or torus, the phase space whose points should label the coherent states, more precisely a cotangent bundle T^*M, where M is the configuration space, is not a homogeneous space. In view of the lack of the general method for the construction of coherent states one is forced to study each case of a particle on a concrete manifold separately. As far as we are aware the most general approach involving the case of the n-dimensional sphere S^n as a configuration space was recently introduced by Hall [7]. In this work we study the coherent states for the quantum mechanics on a torus based on the construction of coherent states for a particle on a circle as a solution of some eigenvalue equation [2]. As a matter of fact, some preliminary results concerning coherent states for the torus utilizing the Zak transform have been described in section 5 of ref. 3. Nevertheless, the exposition of the subject presented herein is much more complete. In section 2 we recall the construction of the coherent states for a quantum particle on a circle [2]. Section 3 deals with the quantum mechanics on a torus. Section 4 is devoted to the definition of the coherent states for the torus and discussion of their basic properties.

II. PRELIMINARIES — COHERENT STATES FOR A QUANTUM PARTICLE ON A CIRCLE

In this section we collect the basic facts about the coherent states for the quantum mechanics on a circle. These states are related to the algebra of the form

$$[J,U]=U,\qquad [J,U^\dagger]=-U^\dagger, \tag{2.1}$$

where J is the angular momentum operator, $U=e^{i\hat{\varphi}}$ is the unitary operator representing the position of a quantum particle on a (unit) circle and we set $\hbar=1$. Consider the eigenvalue equation

$$J|j\rangle = j|j\rangle. \tag{2.2}$$

The operators U and $U^\dagger$ act on vectors $|j\rangle$ as the ladder operators, namely

$$U|j\rangle = |j+1\rangle,\qquad U^\dagger|j\rangle = |j-1\rangle. \tag{2.3}$$

Demanding the time-reversal invariance of the algebra (2.1) we find that the eigenvalues j of the operator J can be only integer and half-integer [2].

The coherent states for the quantum mechanics on a circle can be defined by means of the eigenvalue equation [2]

$$Z|z\rangle = z|z\rangle, \tag{2.4}$$

where

$$Z=e^{-J+\frac{1}{2}}U \tag{2.5}$$

and the complex number

$$z=e^{-l+i\alpha} \tag{2.6}$$

parametrizes the circular cylinder $S^1\times \boldsymbol{R}$ which is the classical phase space for a particle moving in a circle. The coherent states specified by (2.4) can be alternatively obtained by means of the Zak transform [3]. The projection of the vectors $|z\rangle$ onto the basis vectors $|j\rangle$ is given by

$$\langle j|z\rangle = z^{-j}e^{-\frac{j^2}{2}}. \tag{2.7}$$

Using the parameters l, and φ we can write (2.7) as

$$\langle j|l,\alpha\rangle = e^{lj-ij\alpha}e^{-\frac{j^2}{2}}, \tag{2.8}$$

where $|l,\alpha\rangle \equiv |z\rangle$, with $z = e^{-l+i\alpha}$. The coherent states are not orthogonal. We have

$$\langle z|w\rangle = \theta_3\left(\frac{i}{2\pi}\ln z^*w\middle|\frac{i}{\pi}\right), \qquad \text{(integer case)} \tag{2.9a}$$

$$\langle z|w\rangle = \theta_2\left(\frac{i}{2\pi}\ln z^*w\middle|\frac{i}{\pi}\right), \qquad \text{(half-integer case)} \tag{2.9b}$$

where θ_3 and θ_2 are the Jacobi theta-functions defined by

$$\theta_3(v|\tau) = \sum_{n=-\infty}^{\infty} q^{n^2}(e^{i\pi v})^{2n}, \tag{2.10a}$$

$$\theta_2(v|\tau) = \sum_{n=-\infty}^{\infty} q^{(n-\frac{1}{2})^2}(e^{i\pi v})^{2n-1}, \tag{2.10b}$$

where $q = e^{i\pi\tau}$ and $\operatorname{Im}\tau > 0$.

The expectation value of the angular momentum in the coherent state is

$$\frac{\langle l,\alpha|J|l,\alpha\rangle}{\langle l,\alpha|l,\alpha\rangle} \approx l, \tag{2.11}$$

where the very good approximation of the relative error is [2]: $\Delta l/l \approx 2\pi\exp(-\pi^2)\sin(2l\pi)/l$ (see also the very recent paper [8]), so the maximal error arising in the case $l \to 0$ is of order 0.1 per cent. We have remarkable exact equality for l integer or half-integer. Therefore, the parameter l in z can be identified with the classical angular momentum. Furthermore, we have the following formula on the expectation value of the unitary operator U representing the position of a particle on a circle:

$$\frac{\langle l,\alpha|U|l,\alpha\rangle}{\langle l,\alpha|l,\alpha\rangle} \approx e^{-\frac{1}{4}}e^{i\alpha}. \tag{2.12}$$

where the approximation is very good. On defining the relative expectation value

$$\langle\langle U\rangle\rangle_{(l,\alpha)} := \frac{\langle U\rangle_{(l,\alpha)}}{\langle U\rangle_{(0,\alpha)}}, \tag{2.13}$$

where $\langle U\rangle_{(l,\alpha)} = \langle l,\alpha|U|l,\alpha\rangle/\langle l,\alpha|l,\alpha\rangle$, we get

$$\langle\langle U\rangle\rangle_{(l,\alpha)} \approx e^{i\alpha}. \tag{2.14}$$

Therefore, the parameter α can be interpreted as a classical angle. We point out that the approximate relation (2.14) cannot hold in the case of the expectation value $\langle U\rangle_{(l,\alpha)}$ because U is not diagonal in the coherent state basis.

III. QUANTUM MECHANICS ON A TORUS

Now, our experience with the case of the circle discussed in the previous section, in particular the form of the algebra (2.1), and the fact that from the topological point of view the two-torus T^2 can be identified with the product of two circles, indicates the following algebra adequate for the study of the motion on a torus:

$$
\begin{aligned}
&[J_i, U_j] = \delta_{ij} U_j, \qquad [J_i, U_j^\dagger] = -\delta_{ij} U_j^\dagger, \\
&[J_i, J_j] = [U_i, U_j] = [U_i^\dagger, U_j^\dagger] = [U_i, U_j^\dagger] = 0, \qquad i,\, j = 1,\, 2.
\end{aligned}
\tag{3.1}
$$

We point out that a version of the algebra (3.1) satisfied by J_i's and the cosine and sine of the angle operators such that

$$
\cos\hat{\varphi}_i = \tfrac{1}{2}(U_i + U_i^\dagger), \qquad \sin\hat{\varphi}_i = \tfrac{1}{2i}(U_i - U_i^\dagger) \tag{3.2}
$$

were originally introduced in the context of the quantum mechanics on a torus by Isham [9] (see also [10]). Consider the eigenvalue equations

$$
\boldsymbol{J}|\boldsymbol{j}\rangle = \boldsymbol{j}|\boldsymbol{j}\rangle, \tag{3.3}
$$

where $\boldsymbol{J} = (J_1, J_2)$, and $\boldsymbol{j} = (j_1, j_2)$. From (3.1) and (3.3) it follows that the operators U_i and $U_j^\dagger$, $i,\ j\ =\ 1,\ 2$, act on the vectors $|\boldsymbol{j}\rangle$ as the ladder operators, i.e. we have

$$
U_i|\boldsymbol{j}\rangle = |\boldsymbol{j} + \boldsymbol{e}_i\rangle, \qquad U_i^\dagger|\boldsymbol{j}\rangle = |\boldsymbol{j} - \boldsymbol{e}_i\rangle, \tag{3.4}
$$

where $\boldsymbol{e}_1 = (1, 0)$, and $\boldsymbol{e}_2 = (0, 1)$ are the unit vectors. By (3.4) we can generate the whole basis $\{|\boldsymbol{j}\rangle\}$ of the Hilbert space of states from the unique vector $|\boldsymbol{j}_0\rangle$, where $j_{0i} \in [0, 1)$, $i = 1,\ 2$. Evidently, representations with different values of $\boldsymbol{j}_0$ are nonequivalent. Now, let T be the anti-unitary operator of time inversion. We have

$$
TJ_iT^{-1} = -J_i, \qquad TU_iT^{-1} = U_i^\dagger, \qquad TU_i^\dagger T^{-1} = U_i \tag{3.5}
$$

implying the invariance of the algebra (3.1) under time inversion. Further, relations (3.3) and (3.4) imply

$$
T|\boldsymbol{j}\rangle = |-\boldsymbol{j}\rangle. \tag{3.6}
$$

From (3.6) it follows that T is well defined on the Hilbert space of states spanned by the vectors $|\boldsymbol{j}\rangle$ if and only if the spectrum of $\boldsymbol{J}$ is symmetric with respect to $\boldsymbol{0}\ =\ (0, 0)$. In

view of (3.6) this means that $j_{01}=0$ or $j_{01}=\frac{1}{2}$ and $j_{02}=0$ or $j_{02}=\frac{1}{2}$. Clearly $j_{0i}=0$ ($j_{0i}=\frac{1}{2}$) implies integer (half-integer) eigenvalues j_i. Thus, it turns out that demanding the time-reversal invariance we have four possibilities left: j_1-integer and j_2-integer, j_1-integer and j_2-half-integer, j_1-half-integer and j_2-integer, and j_1-half-integer and j_2-half-integer. These cases will be symbolically designated by (0,0), (0,$\frac{1}{2}$), ($\frac{1}{2}$,0) and ($\frac{1}{2}$,$\frac{1}{2}$), respectively, throughout this work. We point out that in opposition to the quantum mechanics on a circle [2] the physical interpretation of the spectrum of the angular momentum operator $\boldsymbol{J}$ is not obvious. For example both cases (0,0) and ($\frac{1}{2}$, $\frac{1}{2}$) seem to correspond to integer spin of a particle, however it is not clear what is the physical difference between them. We finally write down the orthogonality and completeness conditions satisfied by the vectors $|\boldsymbol{j}\rangle$ of the form

$$\langle\boldsymbol{j}|\boldsymbol{j}'\rangle=\delta_{j_1j_1'}\delta_{j_2j_2'}, \tag{3.7}$$

$$\sum_{\boldsymbol{j}\in\boldsymbol{Z}^2}|\boldsymbol{j}\rangle\langle\boldsymbol{j}|=I, \tag{3.8}$$

where $\boldsymbol{Z}$ is the set of integers and the substitution $j_2\to j_2-\frac{1}{2}$, $j_1\to j_1-\frac{1}{2}$, and $j_1\to j_1-\frac{1}{2}$ and $j_2\to j_2-\frac{1}{2}$ in the cases $(0,\frac{1}{2})$, $(\frac{1}{2},0)$ and $(\frac{1}{2},\frac{1}{2})$, respectively, is understood.

We now discuss the coordinate representation for the quantum mechanics on a torus. Consider the common eigenvectors $|\boldsymbol{\varphi}\rangle$ of the operators U_k representing the position of a particle on a torus such that

$$U_k|\boldsymbol{\varphi}\rangle=e^{i\varphi_k}|\boldsymbol{\varphi}\rangle,\qquad k=1,\,2. \tag{3.9}$$

These vectors form the complete set. The resolution of the identity can be written as

$$\frac{1}{4\pi^2}\int\limits_0^{2\pi}d\varphi_1\int\limits_0^{2\pi}d\varphi_2|\boldsymbol{\varphi}\rangle\langle\boldsymbol{\varphi}|=I. \tag{3.10}$$

If we treat torus T^2 as a product of two circles, that is we restrict to the topological aspects of the torus, then completeness gives rise to a functional representation of vectors of the form

$$\langle f|g\rangle=\frac{1}{4\pi^2}\int\limits_0^{2\pi}d\varphi_1\int\limits_0^{2\pi}d\varphi_2 f^*(\boldsymbol{\varphi})g(\boldsymbol{\varphi}), \tag{3.11}$$

where $f(\boldsymbol{\varphi})=\langle\boldsymbol{\varphi}|f\rangle$. Since the basis vectors $|\boldsymbol{j}\rangle$ are represented by the functions

$$e_{\boldsymbol{j}}(\boldsymbol{\varphi})=\langle\boldsymbol{\varphi}|\boldsymbol{j}\rangle=e^{i\boldsymbol{j}\cdot\boldsymbol{\varphi}}, \tag{3.12}$$

where $\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v}=\sum_{i=1}^{2}u_iv_i$, following directly from (3.9) and (3.4), therefore the functions which are the elements of the Hilbert space specified by the scalar product (3.11) are accordingly to (3.12) periodic or antiperiodic ones in φ_1 and φ_2. The operators J_i and U_j, $i,\ j=1,\,2$ act in the representation (3.11) in the following way:

$$J_kf(\boldsymbol{\varphi})=-i\frac{\partial f}{\partial\varphi_k},\qquad U_kf(\boldsymbol{\varphi})=e^{i\varphi_k}f(\boldsymbol{\varphi}),\qquad k=1,\,2. \tag{3.13}$$

We now return to (3.11). An alternative functional representation arises when the torus is viewed as a two-dimensional surface embedded in $\boldsymbol{R}^3$ defined by

$$\begin{aligned} x_1 &= (R+r\cos\varphi_2)\cos\varphi_1, \qquad (3.14)\\ x_2 &= (R+r\cos\varphi_2)\sin\varphi_1,\\ x_3 &= r\sin\varphi_2, \end{aligned}$$

where φ_1 is the azimuthal angle and φ_2 polar angle; R and r are the outer and inner radius of the torus, respectively. In such a case the scalar product is given by [11]

$$\langle\tilde{f}|\tilde{g}\rangle=\frac{1}{4\pi^2}\int\limits_0^{2\pi}d\varphi_1\int\limits_0^{2\pi}d\varphi_2(1+(r/R)\cos\varphi_2)\tilde{f}^*(\boldsymbol{\varphi})\tilde{g}(\boldsymbol{\varphi}), \tag{3.15}$$

where the measure $(1+(r/R)\cos\varphi_2)d\varphi_1d\varphi_2$ coincides up to the multiplicative normalization constant with the surface element of the torus $dS=r(R+r\cos\varphi_2)d\varphi_1d\varphi_2$. The representations (3.11) and (3.15) are isomorphic. The unitary operator mapping (3.11) into (3.15) is of the form

$$Vf(\varphi_1,\varphi_2)=\tilde{f}(\varphi_1,\varphi_2)=\frac{f(\varphi_1,\varphi_2)}{\sqrt{1+(r/R)\cos\varphi_2}}. \tag{3.16}$$

Using (3.16) and (3.13) we find that the operators act in the representation (3.15) as follows

$$\tilde{J}_1\tilde{f}(\boldsymbol{\varphi})=VJ_1V^{-1}\tilde{f}(\boldsymbol{\varphi})=J_1\tilde{f}(\boldsymbol{\varphi})=-i\frac{\partial\tilde{f}}{\partial\varphi_1}, \tag{3.17}$$

$$\tilde{J}_2\tilde{f}(\boldsymbol{\varphi})=VJ_2V^{-1}\tilde{f}(\boldsymbol{\varphi})=-i\frac{\partial\tilde{f}}{\partial\varphi_2}+\frac{i}{2}\frac{(r/R)\sin\varphi_2}{1+(r/R)\cos\varphi_2}\tilde{f}, \tag{3.18}$$

$$\tilde{U}_k\tilde{f}(\boldsymbol{\varphi})=VU_kV^{-1}\tilde{f}(\boldsymbol{\varphi})=U_k\tilde{f}(\boldsymbol{\varphi})=e^{i\varphi_k}\tilde{f},\qquad k=1,\,2. \tag{3.19}$$

We finally point out that the probability density for the coordinates in the normalized state

$|f\rangle$ does not depend on the choice of the representation (3.11) or (3.15). Indeed, we have

$$\frac{1}{4\pi^2}\int\limits_{\varphi_1}^{\varphi_1+\Delta\varphi_1} d\varphi_1 \int\limits_{\varphi_2}^{\varphi_2+\Delta\varphi_2} d\varphi_2(1+(r/R)\cos\varphi_2)|\tilde{f}(\boldsymbol{\varphi})|^2 \tag{3.20}$$

$$= \frac{1}{4\pi^2}\int\limits_{\varphi_1}^{\varphi_1+\Delta\varphi_1} d\varphi_1 \int\limits_{\varphi_2}^{\varphi_2+\Delta\varphi_2} d\varphi_2|f(\boldsymbol{\varphi})|^2 = |f(\boldsymbol{\varphi})|^2\Delta\varphi_1\Delta\varphi_2,$$

where $\Delta\varphi_1 \ll 1$ and $\Delta\varphi_2 \ll 1$.

IV. COHERENT STATES FOR THE QUANTUM MECHANICS ON A TORUS

A. Definition of coherent states

Based on a form of (2.5) we define the coherent states for the quantum mechanics on a torus as the solution of the eigenvalue equation such that

$$\boldsymbol{Z}|\boldsymbol{z}\rangle = \boldsymbol{z}|\boldsymbol{z}\rangle, \tag{4.1}$$

where $\boldsymbol{z} = (z_1, z_2) \in \boldsymbol{C}^2$, and

$$Z_i = e^{-J_i+\frac{1}{2}}U_i, \qquad i = 1,\, 2. \tag{4.2}$$

Taking into account (4.1), (3.3) and (3.4) we get

$$\langle \boldsymbol{j}|\boldsymbol{z}\rangle = z_1^{-j_1} z_2^{-j_2} e^{-\frac{1}{2}\boldsymbol{j}^2}. \tag{4.3}$$

Therefore, the coherent state $|\boldsymbol{z}\rangle$ is given by

$$|\boldsymbol{z}\rangle = \sum_{\boldsymbol{j}\in\boldsymbol{Z}^2} z_1^{-j_1} z_2^{-j_2} e^{-\frac{1}{2}\boldsymbol{j}^2}|\boldsymbol{j}\rangle. \tag{4.4}$$

Using (4.3) and (3.8) we find that the overlap of the coherent states is

$$\langle \boldsymbol{z}|\boldsymbol{w}\rangle = \theta_3\left(\frac{i}{2\pi}\ln z_1^* w_1\middle|\frac{i}{\pi}\right)\theta_3\left(\frac{i}{2\pi}\ln z_2^* w_2\middle|\frac{i}{\pi}\right), \qquad ((0,0)\text{ case}) \tag{4.5a}$$

$$\langle \boldsymbol{z}|\boldsymbol{w}\rangle = \theta_3\left(\frac{i}{2\pi}\ln z_1^* w_1\middle|\frac{i}{\pi}\right)\theta_2\left(\frac{i}{2\pi}\ln z_2^* w_2\middle|\frac{i}{\pi}\right), \qquad ((0,\tfrac{1}{2})\text{ case}) \tag{4.5b}$$

$$\langle \boldsymbol{z}|\boldsymbol{w}\rangle = \theta_2\left(\frac{i}{2\pi}\ln z_1^* w_1\middle|\frac{i}{\pi}\right)\theta_3\left(\frac{i}{2\pi}\ln z_2^* w_2\middle|\frac{i}{\pi}\right), \qquad ((\tfrac{1}{2},0)\text{ case}) \tag{4.5c}$$

$$\langle \boldsymbol{z}|\boldsymbol{w}\rangle = \theta_2\left(\frac{i}{2\pi}\ln z_1^* w_1\middle|\frac{i}{\pi}\right)\theta_2\left(\frac{i}{2\pi}\ln z_2^* w_2\middle|\frac{i}{\pi}\right). \qquad ((\tfrac{1}{2},\tfrac{1}{2})\text{ case}) \tag{4.5d}$$

Now, the phase space for a quantum particle on a torus is the cotangent bundle $T^*T^2 = T^2\times\boldsymbol{R}^2$ which is topologically equivalent to the product of two cylinders $(S^1\times\boldsymbol{R})\times(S^1\times\boldsymbol{R})$. Therefore, we can use the parametrization (2.6) and write the coordinates of the vector $\boldsymbol{z}$ labelling the phase space for a quantum particle on a torus as

$$z_k = e^{-l_k+i\alpha_k},\qquad k=1,\,2. \tag{4.6}$$

The relations (4.3)–(4.5) written in the parametrization (4.6) take the following form:

$$\langle\boldsymbol{j}|\boldsymbol{l},\boldsymbol{\alpha}\rangle = e^{\boldsymbol{l}\cdot\boldsymbol{j}-i\boldsymbol{\alpha}\cdot\boldsymbol{j}}e^{-\frac{1}{2}\boldsymbol{j}^2}, \tag{4.7}$$

where $|\boldsymbol{l},\boldsymbol{\alpha}\rangle\equiv|\boldsymbol{z}\rangle$ with $z_k=e^{-l_k+i\alpha_k}$, $k=1,\,2$. Therefore we can write the coherent state in the form

$$|\boldsymbol{l},\boldsymbol{\alpha}\rangle = \sum_{\boldsymbol{j}\in\boldsymbol{Z}^2} e^{\boldsymbol{l}\cdot\boldsymbol{j}-i\boldsymbol{\alpha}\cdot\boldsymbol{j}}e^{-\frac{1}{2}\boldsymbol{j}^2}|\boldsymbol{j}\rangle. \tag{4.8}$$

An immediate consequence of (4.7), (3.8) and (2.10) are the following formulae on the scalar products (4.5) written in the parametrization (4.6):

$$\langle\boldsymbol{l},\boldsymbol{\alpha}|\boldsymbol{h},\boldsymbol{\beta}\rangle = \theta_3\left(\frac{1}{2\pi}(\alpha_1-\beta_1)-\frac{l_1+h_1}{2}\frac{i}{\pi}\bigg|\frac{i}{\pi}\right)\theta_3\left(\frac{1}{2\pi}(\alpha_2-\beta_2)-\frac{l_2+h_2}{2}\frac{i}{\pi}\bigg|\frac{i}{\pi}\right),\qquad ((0,0)\text{ case}) \tag{4.9a}$$

$$\langle\boldsymbol{l},\boldsymbol{\alpha}|\boldsymbol{h},\boldsymbol{\beta}\rangle = \theta_3\left(\frac{1}{2\pi}(\alpha_1-\beta_1)-\frac{l_1+h_1}{2}\frac{i}{\pi}\bigg|\frac{i}{\pi}\right)\theta_2\left(\frac{1}{2\pi}(\alpha_2-\beta_2)-\frac{l_2+h_2}{2}\frac{i}{\pi}\bigg|\frac{i}{\pi}\right),\qquad ((0,\tfrac{1}{2})\text{ case}) \tag{4.9b}$$

$$\langle\boldsymbol{l},\boldsymbol{\alpha}|\boldsymbol{h},\boldsymbol{\beta}\rangle = \theta_2\left(\frac{1}{2\pi}(\alpha_1-\beta_1)-\frac{l_1+h_1}{2}\frac{i}{\pi}\bigg|\frac{i}{\pi}\right)\theta_3\left(\frac{1}{2\pi}(\alpha_2-\beta_2)-\frac{l_2+h_2}{2}\frac{i}{\pi}\bigg|\frac{i}{\pi}\right),\qquad ((\tfrac{1}{2},0)\text{ case}) \tag{4.9c}$$

$$\langle\boldsymbol{l},\boldsymbol{\alpha}|\boldsymbol{h},\boldsymbol{\beta}\rangle = \theta_2\left(\frac{1}{2\pi}(\alpha_1-\beta_1)-\frac{l_1+h_1}{2}\frac{i}{\pi}\bigg|\frac{i}{\pi}\right)\theta_2\left(\frac{1}{2\pi}(\alpha_2-\beta_2)-\frac{l_2+h_2}{2}\frac{i}{\pi}\bigg|\frac{i}{\pi}\right),\qquad ((\tfrac{1}{2},\tfrac{1}{2})\text{ case}) \tag{4.9d}$$

B. Coherent states and the classical phase space

As with the coherent states $|z\rangle$ for a quantum particle on a circle our criterion to test the correctness of the introduced coherent states $|\boldsymbol{z}\rangle$ for the quantum mechanics on a torus will be their closeness to the classical phase space described by the formulae like (2.11) and (2.14). Consider first the expectation values of the angular momentum $\boldsymbol{J}$. Eqs. (3.8), (3.3)

and (4.7) taken together yield

$$\frac{\langle \boldsymbol{l},\boldsymbol{\alpha}|J_k|\boldsymbol{l},\boldsymbol{\alpha}\rangle}{\langle \boldsymbol{l},\boldsymbol{\alpha}|\boldsymbol{l},\boldsymbol{\alpha}\rangle} = \frac{1}{2\theta_3(\frac{il_k}{\pi}|\frac{i}{\pi})}\frac{d}{dl_k}\theta_3\left(\frac{il_k}{\pi}\middle|\frac{i}{\pi}\right), \quad k=1,\,2, \quad ((0,0)\text{ case}) \tag{4.10a}$$

$$\frac{\langle \boldsymbol{l},\boldsymbol{\alpha}|J_{1(2)}|\boldsymbol{l},\boldsymbol{\alpha}\rangle}{\langle \boldsymbol{l},\boldsymbol{\alpha}|\boldsymbol{l},\boldsymbol{\alpha}\rangle} = \frac{1}{2\theta_{3(2)}(\frac{il_{1(2)}}{\pi}|\frac{i}{\pi})}\frac{d}{dl_{1(2)}}\theta_{3(2)}\left(\frac{il_{1(2)}}{\pi}\middle|\frac{i}{\pi}\right), \qquad ((0,\tfrac{1}{2})\text{ case}) \tag{4.10b}$$

$$\frac{\langle \boldsymbol{l},\boldsymbol{\alpha}|J_{1(2)}|\boldsymbol{l},\boldsymbol{\alpha}\rangle}{\langle \boldsymbol{l},\boldsymbol{\alpha}|\boldsymbol{l},\boldsymbol{\alpha}\rangle} = \frac{1}{2\theta_{2(3)}(\frac{il_{1(2)}}{\pi}|\frac{i}{\pi})}\frac{d}{dl_{1(2)}}\theta_{2(3)}\left(\frac{il_{1(2)}}{\pi}\middle|\frac{i}{\pi}\right), \qquad ((\tfrac{1}{2},0)\text{ case}) \tag{4.10c}$$

$$\frac{\langle \boldsymbol{l},\boldsymbol{\alpha}|J_k|\boldsymbol{l},\boldsymbol{\alpha}\rangle}{\langle \boldsymbol{l},\boldsymbol{\alpha}|\boldsymbol{l},\boldsymbol{\alpha}\rangle} = \frac{1}{2\theta_2(\frac{il_k}{\pi}|\frac{i}{\pi})}\frac{d}{dl_k}\theta_2\left(\frac{il_k}{\pi}\middle|\frac{i}{\pi}\right), \quad k=1,\,2. \quad ((\tfrac{1}{2},\tfrac{1}{2})\text{ case}) \tag{4.10d}$$

Proceeding analogously as in the case of the coherent states for the circle we find that for l_i-integer or half-integer

$$\frac{\langle \boldsymbol{l},\boldsymbol{\alpha}|\boldsymbol{J}|\boldsymbol{l},\boldsymbol{\alpha}\rangle}{\langle \boldsymbol{l},\boldsymbol{\alpha}|\boldsymbol{l},\boldsymbol{\alpha}\rangle} = \boldsymbol{l} \tag{4.11}$$

and in general case

$$\frac{\langle \boldsymbol{l},\boldsymbol{\alpha}|\boldsymbol{J}|\boldsymbol{l},\boldsymbol{\alpha}\rangle}{\langle \boldsymbol{l},\boldsymbol{\alpha}|\boldsymbol{l},\boldsymbol{\alpha}\rangle} \approx \boldsymbol{l}, \tag{4.12}$$

where the approximation is very good (the maximal error is of order 0.1 per cent). Therefore the parameter $\boldsymbol{l}$ can be regarded as a classical angular momentum.

We now discuss the expectation values of the unitary operators U_i representing the position of a quantum particle on a torus. On using (3.8), (3.4) and (4.7) we arrive at the following relations

$$\frac{\langle \boldsymbol{l},\boldsymbol{\alpha}|U_k|\boldsymbol{l},\boldsymbol{\alpha}\rangle}{\langle \boldsymbol{l},\boldsymbol{\alpha}|\boldsymbol{l},\boldsymbol{\alpha}\rangle} = e^{-\frac{1}{4}}e^{i\alpha_k}\frac{\theta_2(\frac{il_k}{\pi}|\frac{i}{\pi})}{\theta_3(\frac{il_k}{\pi}|\frac{i}{\pi})}, \quad k=1,\,2, \quad ((0,0)\text{ case}) \tag{4.13a}$$

$$\frac{\langle \boldsymbol{l},\boldsymbol{\alpha}|U_{1(2)}|\boldsymbol{l},\boldsymbol{\alpha}\rangle}{\langle \boldsymbol{l},\boldsymbol{\alpha}|\boldsymbol{l},\boldsymbol{\alpha}\rangle} = e^{-\frac{1}{4}}e^{i\alpha_{1(2)}}\frac{\theta_{2(3)}(\frac{il_{1(2)}}{\pi}|\frac{i}{\pi})}{\theta_{3(2)}(\frac{il_{1(2)}}{\pi}|\frac{i}{\pi})}, \qquad ((0,\tfrac{1}{2})\text{ case}) \tag{4.13b}$$

$$\frac{\langle \boldsymbol{l},\boldsymbol{\alpha}|U_{1(2)}|\boldsymbol{l},\boldsymbol{\alpha}\rangle}{\langle \boldsymbol{l},\boldsymbol{\alpha}|\boldsymbol{l},\boldsymbol{\alpha}\rangle} = e^{-\frac{1}{4}}e^{i\alpha_{1(2)}}\frac{\theta_{3(2)}(\frac{il_{1(2)}}{\pi}|\frac{i}{\pi})}{\theta_{2(3)}(\frac{il_{1(2)}}{\pi}|\frac{i}{\pi})}, \qquad ((\tfrac{1}{2},0)\text{ case}) \tag{4.13c}$$

$$\frac{\langle \boldsymbol{l},\boldsymbol{\alpha}|U_k|\boldsymbol{l},\boldsymbol{\alpha}\rangle}{\langle \boldsymbol{l},\boldsymbol{\alpha}|\boldsymbol{l},\boldsymbol{\alpha}\rangle} = e^{-\frac{1}{4}}e^{i\alpha_k}\frac{\theta_3(\frac{il_k}{\pi}|\frac{i}{\pi})}{\theta_2(\frac{il_k}{\pi}|\frac{i}{\pi})}, \quad k=1,\,2. \quad ((\tfrac{1}{2},\tfrac{1}{2})\text{ case}) \tag{4.13d}$$

As with the case of the coherent states for the circle it follows that in arbitrary case

$$\frac{\langle \boldsymbol{l},\boldsymbol{\alpha}|U_k|\boldsymbol{l},\boldsymbol{\alpha}\rangle}{\langle \boldsymbol{l},\boldsymbol{\alpha}|\boldsymbol{l},\boldsymbol{\alpha}\rangle} \approx e^{-\frac{1}{4}}e^{i\alpha_k}, \qquad k=1,\,2, \tag{4.14}$$

where the approximation is very good. Therefore, introducing the relative expectation value

$$\langle\!\langle U_k\rangle\!\rangle_{(\boldsymbol{l},\boldsymbol{\alpha})} := \frac{\langle U_k\rangle_{(\boldsymbol{l},\boldsymbol{\alpha})}}{\langle U_k\rangle_{(\boldsymbol{0},\boldsymbol{\alpha})}}, \qquad k=1,\,2, \tag{4.15}$$

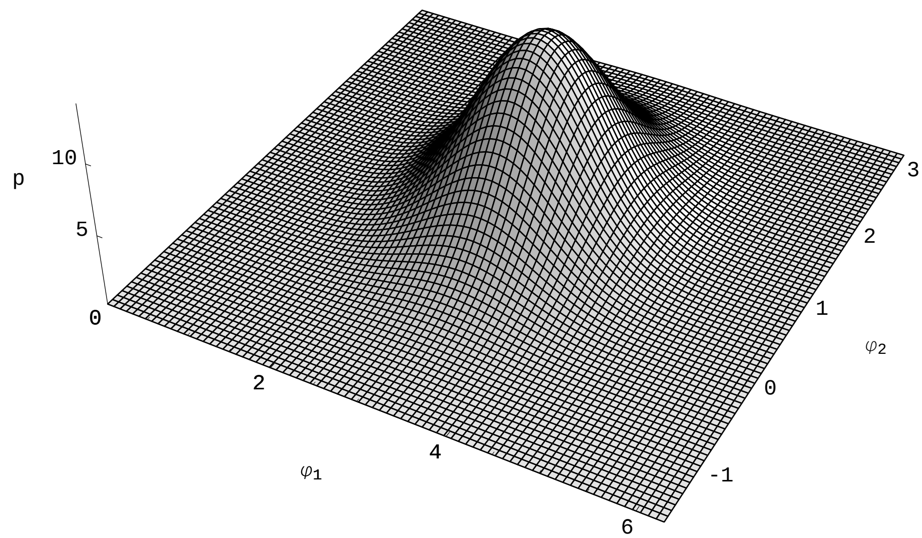

FIG. 1: The plot of the probability density given by (4.17) with $l_1 = 1$, $l_2 = 1$, $\alpha_1 = \pi$ and $\alpha_2 = \pi/3$. The view point slightly above the surface.

where $\langle U_k\rangle_{(\boldsymbol{l},\boldsymbol{\alpha})} = \langle \boldsymbol{l},\boldsymbol{\alpha}|U_k|\boldsymbol{l},\boldsymbol{\alpha}\rangle/\langle \boldsymbol{l},\boldsymbol{\alpha}|\boldsymbol{l},\boldsymbol{\alpha}\rangle$, we obtain

$$\langle\!\langle U_k\rangle\!\rangle_{(\boldsymbol{l},\boldsymbol{\alpha})} \approx e^{i\alpha_k}, \qquad k = 1,\, 2. \tag{4.16}$$

Thus, it turns out that α_k can be regarded as a classical angle parametrizing the position of a particle on a torus. Another evidence for such interpretation of the parameters α_k is provided by the behaviour of the probability density for the coordinates in the normalized coherent state. Indeed, let us restrict for brevity to the case (0,0). The probability density is then given by

$$p_{(\boldsymbol{l},\boldsymbol{\alpha})}(\boldsymbol{\varphi}) = \frac{|\langle \boldsymbol{\varphi}|\boldsymbol{l},\boldsymbol{\alpha}\rangle|^2}{\langle \boldsymbol{l},\boldsymbol{\alpha}|\boldsymbol{l},\boldsymbol{\alpha}\rangle} = \frac{|\theta_3(\frac{1}{2\pi}(\varphi_1-\alpha_1-il_1)\big|\frac{i}{2\pi})\theta_3(\frac{1}{2\pi}(\varphi_2-\alpha_2-il_2)\big|\frac{i}{2\pi})|^2}{\theta_3(\frac{il_1}{\pi}\big|\frac{i}{\pi})\theta_3(\frac{il_2}{\pi}\big|\frac{i}{\pi})} \tag{4.17}$$

following directly from (3.12) and (4.7). From computer simulations it follows that the function $p_{(\boldsymbol{l},\boldsymbol{\alpha})}(\boldsymbol{\varphi})$ is peaked at $\boldsymbol{\varphi} = \boldsymbol{\alpha}$ (see figures). This observation confirms once more the role of α_k as a classical angle.

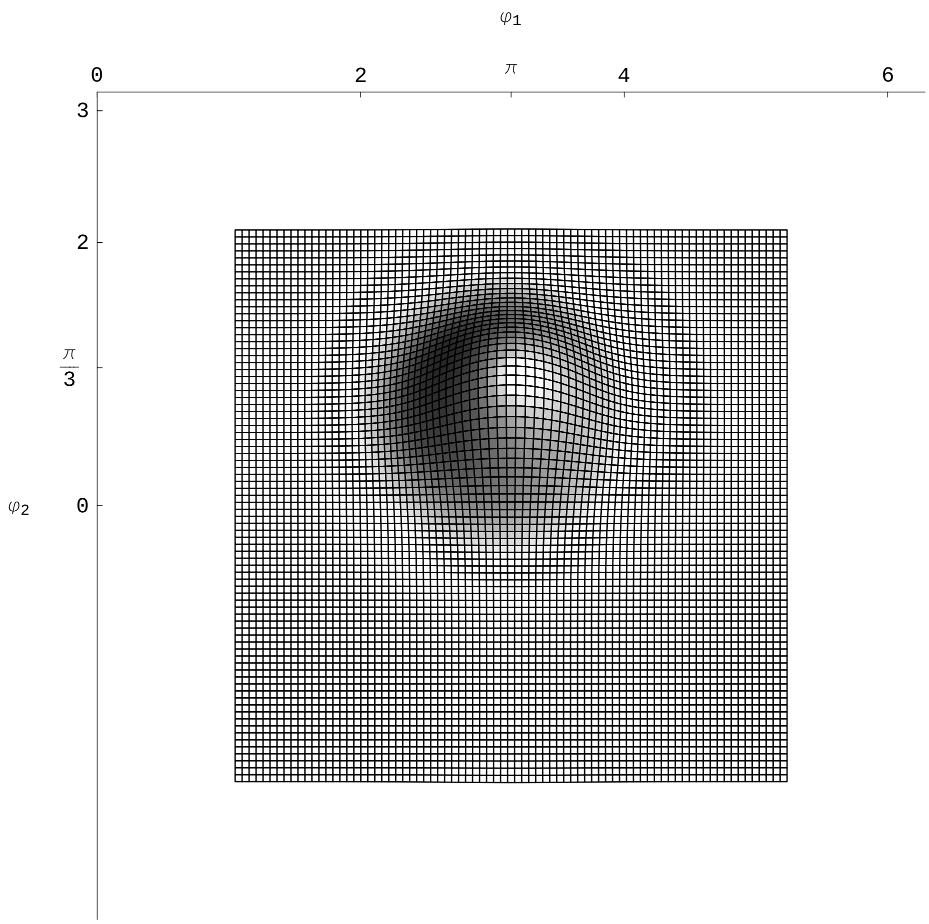

FIG. 2: The plot of the surface from figure 1 with the view point directly above. The maximum of the probability density given by (4.17) at $\varphi_1 = \pi$ and $\varphi_2 = \pi/3$ is easily seen.

C. Generalization to quasiperiodic boundary conditions

Bearing in mind the possible applications of the introduced approach in the condensed matter physics one should involve the case of the general $\boldsymbol{j}_0$ labelling the ground state $|\boldsymbol{j}_0\rangle$ (see (3.4) and discussion below). In fact, the wavefunctions which are the elements of the Hilbert space of states specified by the scalar product (3.11) obey then the general quasiperiodic boundary condition

$$f(\varphi_1, \varphi_2) = e^{i2\pi j_{01}} e^{i2\pi j_{02}} f(\varphi_1, \varphi_2), \tag{4.18}$$

following directly from (3.12) and (3.8) with $\boldsymbol{j}$ replaced with $\boldsymbol{j} + \boldsymbol{j}_0$. Similar functions (Bloch functions) appear in solid state physics. We also point out that the quasiperiodic boundary condition analogous to (4.18) arises for the quantum mechanics on a circle. For the general

case of the quantum mechanics on multiply-connected spaces such condition was discussed by Dürr *et al* [12]. In the context of the quantum mechanics of a free particle on a plane with an extracted point the quasiperiodic boundary condition was investigated by us in [13]. Clearly, the plane with an extracted point is also an example of a multiply-connected space. We recall that the case of the time-reversal symmetry discussed before implied $j_{0i} = 0$ or $j_{0i} = \frac{1}{2}$, $i = 1, 2$, that is the periodic and antiperiodic functions in φ_1 and φ_2. The generalization of the results obtained earlier to the case of arbitrary $\boldsymbol{j}_0$ is straightforward. The scalar product of coherent states is given by

$$\langle \boldsymbol{z}|\boldsymbol{w}\rangle = (z_1^* w_1)^{-j_{01}}(z_2^* w_2)^{-j_{02}} e^{-j_{01}^2-j_{02}^2}\theta_3\left(\frac{i}{2\pi}(\ln z_1^* w_1 + 2j_{01})\middle|\frac{i}{\pi}\right)\theta_3\left(\frac{i}{2\pi}(\ln z_2^* w_2 + 2j_{02})\middle|\frac{i}{\pi}\right). \tag{4.19}$$

The relations (4.5) can be easily obtained from (4.19). Indeed, (4.5a) is an immediate consequence of (4.19). The formulas (4.5b), (4.5c) and (4.5d) referring to $j_{0i} = \frac{1}{2}$, are implied by (4.19) and the identity

$$\theta_{3(2)}(v + \tfrac{\tau}{2}|\tau) = e^{-i\pi(\frac{\tau}{4}+v)}\theta_{2(3)}(v|\tau). \tag{4.20}$$

In the parametrization (4.6) the scalar product of the coherent states takes the form

$$\begin{aligned}\langle \boldsymbol{l},\boldsymbol{\alpha}|\boldsymbol{h},\boldsymbol{\beta}\rangle &= e^{(\boldsymbol{l}+\boldsymbol{h})\cdot\boldsymbol{j}_0 + i(\boldsymbol{\alpha}-\boldsymbol{\beta})\cdot\boldsymbol{j}_0} e^{-\boldsymbol{j}_0^2}\\ &\times\theta_3\left(\frac{1}{2\pi}(\alpha_1-\beta_1) - \frac{l_1+h_1-2j_{01}}{2}\frac{i}{\pi}\middle|\frac{i}{\pi}\right)\theta_3\left(\frac{1}{2\pi}(\alpha_2-\beta_2) - \frac{l_2+h_2-2j_{02}}{2}\frac{i}{\pi}\middle|\frac{i}{\pi}\right).\end{aligned} \tag{4.21}$$

Now, proceeding as with (4.10) we arrive at the following formula on the expectation value of the angular momentum $\boldsymbol{J}$ in the case of an arbitrary $\boldsymbol{j}_0$:

$$\frac{\langle \boldsymbol{l},\boldsymbol{\alpha}|J_k|\boldsymbol{l},\boldsymbol{\alpha}\rangle}{\langle \boldsymbol{l},\boldsymbol{\alpha}|\boldsymbol{l},\boldsymbol{\alpha}\rangle} = j_{0k} + \frac{1}{2\theta_3(\frac{i(l_k-j_{0k})}{\pi}|\frac{i}{\pi})}\frac{d}{dl_k}\theta_3\left(\frac{i(l_k-j_{0k})}{\pi}\middle|\frac{i}{\pi}\right), \quad k = 1, 2. \tag{4.22}$$

From (4.22) and (4.10a) it follows that the approximate relation (4.12) holds in the case of an arbitrary $\boldsymbol{j}_0$. It can be easily checked that (4.10) are implied by (4.22), (4.20) and the fact that the Jacobi theta-functions are even functions of v, i.e. $\theta_3(-v|\tau) = \theta_3(v|\tau)$, and $\theta_2(-v|\tau) = \theta_2(v|\tau)$. Furthermore, using technique applied for derivation (4.13) we get the expectation values of the unitary operators U_i representing the position of the quantum particle on a torus such that

$$\frac{\langle \boldsymbol{l},\boldsymbol{\alpha}|U_k|\boldsymbol{l},\boldsymbol{\alpha}\rangle}{\langle \boldsymbol{l},\boldsymbol{\alpha}|\boldsymbol{l},\boldsymbol{\alpha}\rangle} = e^{-\frac{1}{4}}e^{i\alpha_k}\frac{\theta_2(\frac{i}{\pi}(l_k-j_{0k})|\frac{i}{\pi})}{\theta_3(\frac{i}{\pi}(l_k-j_{0k})|\frac{i}{\pi})}, \quad k = 1, 2. \tag{4.23}$$

The formulas (4.13) can be easily derived from (4.23) with the help of (4.20). Taking into account (4.23) and (4.13a) we find that the approximation relation (4.14) is valid for every $\boldsymbol{j}_0$. Finally, one can easily obtain the following generalization of the formula (4.17) on the probability density for the coordinates in the normalized coherent state

$$p_{(\boldsymbol{l},\boldsymbol{\alpha})}(\boldsymbol{\varphi}) = \frac{|\langle\boldsymbol{\varphi}|\boldsymbol{l},\boldsymbol{\alpha}\rangle|^2}{\langle\boldsymbol{l},\boldsymbol{\alpha}|\boldsymbol{l},\boldsymbol{\alpha}\rangle} = \frac{|\theta_3(\frac{1}{2\pi}(\varphi_1-\alpha_1-i(l_1-j_{01}))\big|\frac{i}{2\pi})\theta_3(\frac{1}{2\pi}(\varphi_2-\alpha_2-i(l_2-j_{02}))\big|\frac{i}{2\pi})|^2}{\theta_3(\frac{i}{\pi}(l_1-j_{01})\big|\frac{i}{\pi})\theta_3(\frac{i}{\pi}(l_2-j_{02})\big|\frac{i}{\pi})}. \tag{4.24}$$

Comparing (4.24) and (4.17) we find that the probability density (4.24) is peaked at $\boldsymbol{\varphi} = \boldsymbol{\alpha}$ for an arbitrary $\boldsymbol{j}_0$.

V. CONCLUSION

In this work we have introduced the coherent states for the quantum mechanics on a torus. We have not discussed in this paper the Bargmann representation which can be easily obtained by using the relations derived for the circle [2]. In our opinion, besides of the possible applications in the solid state physics mentioned above, the observations of this work would be of importance in the theory of quantum chaos. We only recall that the torus is the configuration space of the double pendulum and toroidal pendulum which are well known to show chaotic behaviour. Another possible application of the constructed coherent states is nanotechnology, especially nanoscopic quantum rings [14]. Furthermore, we hope that similarly as with the coherent states for a particle on a circle which have been applied by Ashtekar *et al* [15] in loop quantum gravity also introduced coherent states for the torus would be of importance in this theory. We finally remark that the results of this paper can be immediately generalized to the case of the n-dimensional torus.

Acknowledgements

This paper has been supported by the Polish Ministry of Scientific Research and Information Technology under the grant No PBZ-MIN-008/P03/2003.

---

[1] J.R. Klauder and B.S. Skagerstam *Coherent States–Applications in Physics and Mathematical Physics* (World Scientific, Singapore, 1985).

[2] K. Kowalski *et al*, J. Phys. A **29**, 4149 (1996).

[3] J.A. Gonzáles and M.A. del Olmo, J. Phys. A **31**, 8841 (1998).

[4] K. Kowalski and J. Rembieliński, J. Phys. A **33**, 6035 (2000) ; K. Kowalski and J. Rembieliński, J. Math. Phys. **42**, 4138 (2001).

[5] H.A. Kastrup, Phys. Rev. A **73**, 052104 (2006).

[6] A.M. Perelomov, *Generalized Coherent States and Their Applications* (Springer, Berlin, 1986).

[7] B.C. Hall and J.J. Mitchell, J. Math. Phys. **43**, 1211 (2002).

[8] B. Bahr and H.J. Korsch, Quantum mechanics on a circle: Husimi phase space distributions and semiclassical coherent states propagators, quant-ph/0611116 (2006).

[9] C.J. Isham, Topological and global aspects of quantum theory, in: DeWitt B S and Stora R, editors, *Relativity, Groups and Topology II*,Les Houches, Session XL. 1983 (Elsevier, Amsterdam, 1984), pp. 1059–1290.

[10] H. Zainuddin, Phys. Rev. D **40**, 636 (1989).

[11] S. Ishikawa *et al*, Int. J. Mod. Phys. A **11**, 3363 (1996).

[12] D. Dürr *et al*, J. Phys. A **40**, 2997 (2007); D. Dürr *et al*, Ann. Inst. H. Poincaré Phys. Theor. **7**, 791 (2006).

[13] K. Kowalski *et al*, Phys. Rev. A **66**, 032118 (2002).

[14] A. Lorke *et al*, Phys. Rev. Lett. **84**, 2223 (2000).

[15] A. Ashtekar *et al*, Class. Quantum Grav. **20**, 1031 (2003).